

On 3-dimensional asymptotically harmonic manifolds

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Abstract. Let (M, g) be a complete, simply connected Riemannian manifold of dimension 3 without conjugate points. We show that M is a hyperbolic manifold of constant sectional curvature $-\frac{h^2}{4}$, provided M is asymptotically harmonic of constant $h > 0$.

Mathematics Subject Classification (2000). Primary 53C35; Secondary 53C25.

Keywords. Asymptotic harmonic manifold, horospheres.

1. Introduction. Let (M, g) be a complete, simply connected Riemannian manifold without conjugate points. Let SM be the unit tangent bundle of M . For $v \in SM$, let γ_v be the geodesic with $\gamma_v'(0) = v$ and $b_{v,t}(x) = \lim_{t \rightarrow \infty} (d(x, \gamma_v(t)) - t)$ the corresponding *Busemann function* for γ_v . The level sets $b_v^{-1}(t)$ are called *horospheres*.

A complete, simply connected Riemannian manifold without conjugate points is called *asymptotically harmonic* if the mean curvature of its horospheres is a universal constant, that is if its Busemann functions satisfy $\Delta b_v \equiv h$, $\forall v \in SM$, where h is a nonnegative constant. Then b_v is a smooth function on M for all v and all horospheres of M are smooth, simply connected hypersurfaces in M with constant mean curvature h .

For example, every simply connected, complete harmonic manifold without conjugate points is asymptotically harmonic.

For more details on this subject we refer to the discussion and to the references in [2]. Important result in this context are contained in [1], [3]. In [2] the following result was proved:

*Supported by Swiss National Science Foundation.

†The author thanks Forschungsinstitut für Mathematik, ETH Zürich for its hospitality and support.

Let M be a Hadamard manifold of dimension 3 whose sectional curvatures are bounded from above by a negative constant (i.e. $K \leq -a^2$ for some $a \neq 0$) and whose curvature tensor satisfies $\|\nabla R\| \leq C$ for a suitable constant C . If M is asymptotically harmonic, then M is symmetric and hence of constant sectional curvature.

We prove this result without any hypothesis on the curvature tensor.

Theorem 1.1. *Let (M, g) be a complete, simply connected Riemannian manifold of dimension 3 without conjugate points. If M is asymptotically harmonic of constant $h > 0$, then M is a manifold of constant sectional curvature $\frac{-h^2}{4}$.*

2. Proof of the Theorem. The first part of the proof (Lemma 2.1 to Lemma 2.3) is a modification of the results in [2]. Therefore we recall some notations which were already used in that paper. Our general assumption is that M is 3-dimensional, has no conjugate points and is asymptotically harmonic with constant $h > 0$. For $v \in SM$ and $x \in v^\perp$, let

$$u^+(v)(x) = \nabla_x \nabla b_{-v} \quad \text{and} \quad u^-(v)(x) = -\nabla_x \nabla b_v.$$

Thus $u^\pm(v) \in \text{End}(v^\perp)$. With $\lambda_1(v), \lambda_2(v)$ we denote the eigenvalues of $u^+(v)$. The endomorphism fields u^\pm satisfy the Riccati equation along the orbits of the geodesic flow $\varphi^t : SM \rightarrow SM$.

Thus if $u^\pm(t) := u^\pm(\varphi^t v)$ and $R(t) := R(\cdot, \gamma'_v(t))\gamma'_v(t) \in \text{End}(\gamma'_v(t)^\perp)$, then

$$(u^\pm)' + (u^\pm)^2 + R = 0.$$

We define $V(v) = u^+(v) - u^-(v)$ and correspondingly $V(t) = V(\varphi^t(v))$ along $\gamma_v(t)$. We also define $X(v) = \frac{-1}{2}(u^+(v) + u^-(v))$ and $X(t) = X(\varphi^t(v))$. Then the Riccati equation for $u^\pm(t)$ yields

$$(1) \quad XV + VX = (u^-)^2 - (u^+)^2 = (u^+)' - (u^-)' = V'.$$

Lemma 2.1. *For fixed $v \in SM$ the map $t \mapsto \det V(\varphi^t v)$ is constant.*

Proof. Assume that $V(t)$ is invertible, then

$$\frac{d}{dt} \log \det V(t) = \text{tr } V'(t)V^{-1}(t) = \text{tr } (XV + VX)V^{-1}(t) = 2 \text{tr } X = 0.$$

The last step follows as M is asymptotically harmonic. Thus as long as $\det V(t) \neq 0$, it is constant along γ_v . Therefore $\det V(t)$ is constant along γ_v in any case. \square

Lemma 2.2. *Let $v \in SM$ be such that $V(v) = \mu \text{Id}$, for some $\mu \in \mathbb{R}$, then $R(t) = \frac{-h^2}{4} \text{Id}$, $\forall t$.*

Proof. Note that if $V(v) = \mu \text{Id}$, then $V(\gamma'_v(t)) = h \text{Id}$ for all t , as $\text{tr } V \equiv 2h$ and by Lemma 2.1 the determinant of V is constant along $\gamma_v(t)$. Now by equation (1) $V' = XV + VX$. Hence, along γ_v , $V'(t) \equiv 0$. Thus $2hX = 0$ and since we assume $h > 0$ we have $X = 0$ along γ_v . Therefore, $u^+(t) = -u^-(t)$. But from the

definition of V , $u^+(t) \equiv \frac{h}{2} \text{Id}$ i.e. u^+ is a scalar operator. By the Riccati equation $(u^+(t))^2 + R(t) = 0$, i.e. $R(t) = -\frac{h^2}{4} \text{Id}$. \square

Lemma 2.3. *For every point $p \in M$ there exists $v \in S_p M$ such that $R(x, v)v = -\frac{h^2}{4} x$, $\forall x \in v^\perp$. In particular, $\text{Ric}(v, v) = -\frac{h^2}{2}$.*

Proof. Since TS^2 is nontrivial, an easy topological argument shows, that for every $p \in M$ there exists $v \in S_p M$ such that the two eigenvalues of $V(v)$ coincide. Thus $V(v) = \mu \text{Id}$. The result now follows from Lemma 2.2. \square

Lemma 2.4. *For all $v \in SM$ we have $\text{Ric}(v, v) \leq -\frac{h^2}{2}$.*

Proof. The Riccati equation for $t \mapsto u^+(t)$ implies $(u^+)' + (u^+)^2 + R = 0$. Hence, $\text{tr}(u^+)^2 + \text{tr} R = 0$. Thus, $\text{Ric}(v, v) = -(\lambda_1^2(v) + \lambda_2^2(v))$. By hypothesis $\lambda_1(v) + \lambda_2(v) = h$, hence $\lambda_1^2(v) + \lambda_2^2(v) \geq \frac{h^2}{2}$. Consequently, $\text{Ric}(v, v) \leq -\frac{h^2}{2}$. \square

Lemma 2.5. *The sectional curvature K of M satisfies $K \leq -\frac{h^2}{4}$.*

Proof. Let $p \in M$, and let v be the vector in Lemma 2.3. Take $e_1 = v$, and let e_2 and e_3 be unit vectors orthogonal to e_1 so that $\{e_1, e_2, e_3\}$ forms an orthonormal basis of $T_p M$. Then $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$ forms an orthonormal basis of $\Lambda^2 T_p M$. We want to show that the curvature operator, considered as map $R : \Lambda^2 T_p M \rightarrow \Lambda^2 T_p M$, $\langle R(X \wedge Y), V \wedge W \rangle = \langle R(X, Y)W, V \rangle$ is diagonal in this basis.

From Lemma 2.3 we see $R(e_2, e_1)e_1 = -\frac{h^2}{4} e_2$, $R(e_3, e_1)e_1 = -\frac{h^2}{4} e_3$. Thus $K(e_1, e_2) = K(e_1, e_3) = -\frac{h^2}{4}$ and $K(e_2, e_3) \leq -\frac{h^2}{4}$ as $\text{Ric}(e_3, e_3) \leq -\frac{h^2}{2}$, where $K(v, w)$ denotes the sectional curvature of the plane spanned by v and w . We will prove below that

$$(2) \quad \langle R(e_1, e_3)e_3, e_2 \rangle = 0 \quad \text{and} \quad \langle R(e_1, e_2)e_2, e_3 \rangle = 0.$$

Assuming this for a moment, it follows that $R(e_1 \wedge e_3) \perp \text{span}\{e_1 \wedge e_2, e_2 \wedge e_3\}$ and $R(e_1 \wedge e_2) \perp \text{span}\{e_1 \wedge e_3, e_2 \wedge e_3\}$. Hence,

$$R(e_1 \wedge e_2) = -\frac{h^2}{4} e_1 \wedge e_2 \quad \text{and} \quad R(e_1 \wedge e_3) = -\frac{h^2}{4} e_1 \wedge e_3.$$

Since $e_1 \wedge e_2$ and $e_1 \wedge e_3$ are eigenvectors of R , also $e_2 \wedge e_3$ is an eigenvector and we obtain

$$R(e_2 \wedge e_3) = K(e_2, e_3) e_2 \wedge e_3.$$

Thus the curvature operator is diagonal in the basis $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$ and all eigenvalues are $\leq -\frac{h^2}{4}$, which proves the result.

It remains to show (2). Consider for $t \in (-\varepsilon, \varepsilon)$ the vectors $v_t = \cos t e_1 + \sin t e_2$. Then,

$$\begin{aligned} f(t) &:= \text{Ric}(v_t, v_t) = K(v_t, e_3) + K(v_t, -e_1 \sin t + e_2 \cos t) \\ &= K(e_1, e_2) + \sin^2 t K(e_2, e_3) + \cos^2 t K(e_1, e_3) + \sin 2t \langle R(e_1, e_3)e_3, e_2 \rangle. \end{aligned}$$

By Lemma 2.4 $f(0) = \text{Ric}(v, v) = \frac{-h^2}{2}$ is maximal and hence $f'(0) = 0$. This implies the first equation in (2). If we replace e_2 by e_3 in the above computation we obtain the second equation. \square

Finally we come to the

Proof of Theorem 1.1. Lemma 2.5 implies that $K_M \leq \frac{-h^2}{4}$. By standard comparison geometry we obtain $\lambda_1(v) \geq \frac{h}{2}$ and $\lambda_2(v) \geq \frac{h}{2}$. Now $\lambda_1 + \lambda_2 = h$ implies that $\lambda_1 = \lambda_2 = \frac{h}{2}$. Hence, $u^+(v)$ is a scalar operator and therefore $R(x, v)v = \frac{-h^2}{4}x$, $\forall v$ and $\forall x \in v^\perp$. Thus, $K_M \equiv \frac{-h^2}{4}$.

3. Final Remark. We expect that the result holds also in the case $h = 0$, i.e. if all horospheres are minimal. Our argument, however, uses $h > 0$ essentially in the proof of Lemma 2.2.

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Received: 4 October 2007